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Free convection oscillatory flow along an infinite vertical plate with constant suction

By Krishna Lal

Banaras Hindu University, Varanasi-5, India

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Similarity solution using spiral group transformations has been obtained. For constant suction at the wall, the velocity and temperature of the wall have been found to vary according to $\exp(pt)$, p being certain constant. The equations of motion and energy have been linearized and solved. For small frequency of oscillation, the skin friction is found to be constant after some time. The rate of heat transfer from the plate to the fluid has a phase lead over the surface temperature fluctuations for small and large frequency of oscillations.

Basic Equations

Taking $\bar{x}$ axis vertically along the plate and $\bar{y}$ axis perpendicular to it, the basic equations which describe the unsteady free convection flow of a viscous incompressible fluid of density ρ past an infinite flat plate are

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \qquad \text{.. (1.1)}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g\beta\,(\bar{T} - \bar{T}_\infty) + \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{.. (1.2)}$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{y}} \qquad \text{...(1.3)}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = k\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \qquad \text{.. (1.4)}$$

where $\bar{u}$, $\bar{v}$ are the velocity components, $\bar{T}$ is the temperature, g the acceleration due to gravity, β is the coefficient of volume expansion, k is the thermal diffusivity, $\bar{p}$ is the pressure, $\bar{T}_\infty$ is the equilibrium temperature.

We introduce dimensionless quantities

$$\left.\begin{aligned} &y = \bar{y}/L,\ t = \nu\bar{t}/L^2,\ u = \frac{\bar{u}L}{\nu},\ v = \frac{\bar{v}L}{\nu} \\ &T = g\beta L^3(\bar{T} - \bar{T}_\infty)/\nu^2,\ p = \bar{p}L^2/\rho\nu^2 \end{aligned}\right\} \qquad \text{...(1.5)}$$

where L is the characteristic length. Thus we have from above equations

$$\frac{\partial v}{\partial y} = 0 \qquad \text{...(1.6)}$$

$$\frac{\partial u}{\partial t}+v\,\frac{\partial u}{\partial y}=T+\frac{\partial^2 u}{\partial y^2} \qquad \text{...(1.7)}$$

$$\frac{\partial v}{\partial t}=-\frac{\partial p}{\partial y} \qquad \text{...(1.8)}$$

$$\frac{\partial T}{\partial t}+v\,\frac{\partial T}{\partial y}=\frac{1}{\sigma}\,\frac{\partial^2 T}{\partial y^2} \qquad \text{...(1.9)}$$

where σ is the Prandtl number.

The boundary conditions are

$$\left.\begin{array}{l} y=0,\, u=0,\, v=v_0 f(t),\; T=1+\epsilon\cos\omega t,\; \epsilon\ll 1 \\ y=\infty, \quad u=0, \quad T=0 \end{array}\right\} \qquad \text{...(1.10)}$$

Similarity Solutions by Spiral Transformations

To get the similar solutions of the equations (1.7) and (1.9), we introduce

$$t=\bar{t}+\beta_1 b,\; u=e^{\beta_2 b}\,\bar{u}, v=e^{\beta_3 b}\,\bar{v},\; y=e^{\beta_4 b}\,\bar{y},\; T=e^{\beta_5 b}\,\bar{T} \qquad \text{...(1.11)}$$

into above equations, where $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ and b are certain constants. Thus we have following relations between the constants

$$\left.\begin{array}{l} \beta_3=\beta_2+\beta_3-\beta_4=\beta_5=\beta_2-2\beta_4 \\ \beta_5=\beta_3+\beta_5-\beta_4=\beta_5-2\beta_4 \end{array}\right\} \qquad \text{...(1.12)}$$

which give

$$\beta_2=\beta_5, \quad \beta_3=\beta_4=0 \qquad \text{...(1.13)}$$

Substituting from equations (1.13) into (1.11), we see that

$$\left.\begin{array}{l} \dfrac{u}{e^{pt}}=\dfrac{\bar{u}}{e^{p\bar{t}}} \\ v=v_0 \text{ (constant)} \\ y=\bar{y} \\ \dfrac{T}{e^{pt}}=\dfrac{\bar{T}}{e^{p\bar{t}}} \end{array}\right\} \qquad \text{...(1.14)}$$

where $p=\beta_2/\beta_1=$ constant.

And the absolute invariants are

$$\left.\begin{array}{l} u=F(y)e^{pt} \\ T=\theta(y)e^{pt}, \\ v=-v_0 \text{ (for suction)} \end{array}\right\} \qquad \text{...(1.15)}$$

for similarity solutions

METHOD OF SOLUTION

In solving equations (1.7) and (1.9) for constant suction at the wall, we replace the first boundary condition for T as

$$T = 1 + \epsilon\, e^{i\omega t} \qquad ...2.1$$

and replace u, v and T by

$$\left.\begin{aligned} u &= u_s + \epsilon u_1 e^{i\omega t} \\ v &= -v_0 \text{ (constant)} \\ T &= T_s + \epsilon T_1 e^{i\omega t} \end{aligned}\right\} \qquad ...(2.2)$$

in above equations equations where u_s, T_s the steady mean flow satisfy

$$\left.\begin{aligned} -v_0 \frac{du_s}{dy} &= T_s + \frac{d^2u_s}{dy^2} \\ -v_0 \frac{dT_s}{dy} &= \frac{1}{\sigma} \frac{d^2T_s}{dy^2} \end{aligned}\right\} \qquad ...(2.3)$$

with boundary conditions

$$\left.\begin{aligned} y = 0 &: u_s = 0,\ T_s = 1 \\ y = \infty &: u_s = 0,\ T_s = 0 \end{aligned}\right\}$$

and equating the coefficient of $\epsilon e^{i\omega t}$, we find that u_1 and T_1 satisfy

$$\left.\begin{aligned} i\omega u_1 - v_0 \frac{du_1}{dy} &= T_1 + \frac{d^2u_1}{dy^2} \\ i\omega T_1 - v_0 \frac{dT_1}{dy} &= \frac{1}{\sigma} \frac{d^2T_1}{dy^2} \end{aligned}\right\} \qquad ...(2.5)$$

With the boundary conditions

$$\left.\begin{aligned} y = 0 &: u_1 = 0,\ T_1 = 1 \\ y = \infty &: u_1 = 0,\ T_1 = 0 \end{aligned}\right\} \qquad ..(2.6)$$

Solving above equations we have

$$u_s = \frac{1}{v_0^2(\sigma - \sigma^2)} [\exp(-\sigma v_0 y) - \exp(-v_0 y)] \qquad ..(2.7)$$

$$T_s = e^{-\sigma v_0 y} \qquad ...(2.8)$$

$$T_1 = e^{-hy} \qquad ..(2.9)$$

$$u_1 = \frac{1}{h^2 - v_0 h - i\omega} [\exp(-Sy) - \exp(-hy)] \qquad ...(2.10)$$

where $$h = \frac{v_0\sigma}{2}\left[1 + \left(1 + \frac{4i\omega}{v_0^2\sigma}\right)^{1/2}\right] \qquad ...(2.11)$$

and $$S = \frac{v_0}{2}\left[1 + \left(1 + \frac{4i\omega}{v_0^2}\right)^{1/2}\right] \qquad .(2.12)$$

Discussions

(i) When frequency is small

Expanding for h and S, in powers of ω, we find that

$$h \simeq \sigma v_0 + \frac{i\omega}{v_0} + \frac{\omega^2}{v_0^3 \sigma} + 0(\omega^3) \qquad \text{...(3.1)}$$

$$S \simeq v_0 + \frac{i\omega}{v_0} + \frac{\omega^2}{v_0^3} + 0(\omega^3) \qquad \text{.. (3.2)}$$

Thus, we get for u_1 and T_1

$$u_1 = \frac{1}{v_0^2(\sigma^2 - \sigma)} \Bigg\{ \left(e^{-v_0 y} - e^{-v_0 \sigma} \right) + \frac{i\omega}{v_0} \Bigg[y \left(e^{-v_0 \sigma y} - e^{-v_0 y} \right)$$

$$+ \frac{2(\sigma - v_0)}{v_0^2(\sigma^2 - \sigma)} \left(e^{-v_0 \sigma y} - e^{-v_0 y} \right) \Bigg] + \frac{\omega^2}{v_0^2} \Bigg[\frac{(e^{-v_0 \quad y} - e^{-v_0 y})}{v_0^2 \sigma^2}$$

$$+ \frac{4(\sigma - v_0)^2}{v_0^2(\sigma^2 - \sigma)^2} \left(e^{-v_0 \sigma y} - e^{-v_0 y} \right) + \frac{2y(\sigma - v_0)}{v_0^2(\sigma^2 - \sigma)}$$

$$+ \left(e^{-v_0 \sigma y} - e^{-v_0 y} \right) + \frac{y}{\sigma v_0} \left(e^{-v_0 \sigma y} - e^{-v_0 y} \right)$$

$$+ \frac{y^2}{2} \left(e^{-v_0 \sigma y} - e^{-v_0 y} \right) \Bigg] \Bigg\} \qquad \text{.. (3.3)}$$

$$T_1 = e^{-v_0 \sigma y} \left\{ 1 - \frac{i\omega y}{v_0} - \frac{\omega^2}{v_0^2} \left(\frac{y}{v_0 \sigma} + \frac{y^2}{2} \right) + \dots \right\} \qquad \text{.. (3.4)}$$

From above expressions (3.3) and (3.4) we see that the solutions for u_s and T_s as given by the equations (2.7) and (2.8) are obtained by putting $\omega = 0$.

The velocity field in the boundary layer is given by

$$u(y, t) = \frac{1}{v_0^2(\sigma - \sigma^2)} \left[e^{-\sigma v_0 y} - e^{-v_0 y} \right] + \frac{\epsilon e^{i\omega t}[e^{-Sy} - e^{-hy}]}{h^2 - v_0 h - i\omega} \qquad \text{...(3.5)}$$

The skin friction τ_0 is given by

$$\tau_0 = \mu \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y} = 0}$$

$$= \frac{\mu \nu}{L^2} \left[\frac{1}{\sigma v_0} + \frac{h - S}{h^2 - h v_0 - i\omega} \epsilon e^{i\omega t} \right] \qquad \text{.. (3.6)}$$

Thus the non-dimensional form of τ_0 defined by τ'_0 is given by

$$\tau'_0 = \frac{\tau_0 L^2}{\mu \nu} = \frac{1}{\sigma v_0} + \epsilon \mid \beta \mid \cos(\omega t + \alpha) \qquad \text{...(3.7)}$$

$$\left.\begin{aligned} &\text{where} \quad B = B_r + i\,B_i = \frac{h - S}{h^2 - v_0 h - i\omega} \\ &\text{and} \quad \alpha = \tan^{-1} \frac{B_i}{B_r} \end{aligned}\right\} \qquad \ldots(3.8)$$

For small value of ω, we see that

$$\frac{h - S}{h^2 - v_0 h - i\omega} = \left\{\frac{1}{\sigma v_0} - \frac{\omega^2}{\sigma^2 v_0^5}\left[1 + \frac{4(\sigma - v_0)^2}{\sigma(\sigma - 1)^2 v_0^2}\right]\right\} + i\left\{-\frac{2\omega(\sigma - v_0)}{v_0^4 \sigma^2(\sigma - 1)}\right\} \qquad \ldots(3.9)$$

Thus for $\sigma > v_0$, we see that

$$\tau'_0 = \frac{1}{\sigma r_0} + \epsilon \mid B \mid \cos(\omega t - \alpha) \qquad \ldots(3.10)$$

and for

$$\omega t - \alpha = \pi(n + \tfrac{1}{2}),\ n = 0, 1, 2, \qquad \ldots(3.11)$$

the skin-friction becomes independent of time.

Hence after time t, given by

$$\frac{1}{\omega}[\alpha + \pi(n + \tfrac{1}{2})]$$

the skin friction is independent of time and remains a constant quantity.

The rate of heat transfer from the wall to the fluid is

$$q = -k\left(\frac{d\bar{T}}{d\bar{y}}\right)_{\bar{y}=0} = \frac{k\nu^2}{L^4 g\beta}(\sigma v_0 + h\epsilon e^{i\omega t}) \qquad \ldots(3.12)$$

Defining non-dimensional quantity q' by

$$q' = qL^4 g\beta / k\nu^2,$$

we have $\quad q' = \sigma v_0 + \epsilon e^{i\omega t}(h_r + ih_i) \qquad \ldots(3.13)$

where

$$h = h_r + ih_i = \left(v_0\sigma + \frac{\omega^2}{v_0^3\sigma}\right) + i\left(\frac{\omega}{v_0}\right)$$

for small frequency of oscillations.

Thus

$$q' = \sigma v_0 + E \mid h \mid \cos(\omega t + \beta) \qquad \ldots(3.14)$$

$$\text{where } \beta = \tan^{-1}\frac{h_i}{h_r}$$

and the rate of heat transfer from the plate to the fluid has a phase lead over the surface temperature fluctuations. This phase lead, β increases as ω decreases for given v_0 and σ.

(ii) *When ω is large*

In this case we may write

$$\left.\begin{aligned} h &\simeq \sqrt{\frac{i\omega}{\sigma}} \\ s &\simeq \sqrt{i\omega} \end{aligned}\right\} \qquad \ldots 3.15$$

and thus

$$T_1 = e^{-y\sqrt{i\omega/\sigma}} \qquad \ldots 3.16$$

$$u_1 = \frac{1}{i\omega\left(\frac{1-\sigma}{\sigma}\right) - v_0\sqrt{\frac{i\omega}{\sigma}}}\left[e^{-y\sqrt{i\omega}} - e^{-y\sqrt{i\omega/\sigma}}\right] \qquad . \ 3.17$$

$$\tau_0' = \frac{1}{\sigma v_0} + E \mid A \mid \cos(\omega t + \gamma) \qquad \ldots 3.18$$

$$\left.\begin{aligned} \text{where } A = A_r + i\,A_i &= \frac{\sqrt{\sigma} - \sigma}{\sqrt{i\omega}\,(1-\sigma) - v_0\sqrt{\sigma}} \\ r &= \tan^{-1}\frac{A_i}{A_r} \end{aligned}\right\} \qquad \ldots 3.19$$

We find that γ is a positive quantity and thus τ'_0 decreases as γ increases. Similarly, we see that

$$q' = \sigma v_0 + \epsilon h e^{i\omega t} = \sigma v_0 + \epsilon \mid l \mid \cos(\omega t + \delta) \qquad \ldots 3{\cdot}20$$

where

$$l = l_r + i\,l_i + \sqrt{\frac{i\omega}{\sigma}}\,, \ \delta = \tan^{-1}\frac{l_i}{l_r}$$

Thus $\quad \delta = \pi/u, \ \mid l \mid = \sqrt{\omega/\sigma}$

In this case the rate of heat transfer from the plate to the fluid has a phase lead over the surface temperature fluctuations by an angle $\frac{\pi}{4}$

Conclusion

For small frequency of oscillations, the skin friction lags behind by a certain angle while for large frequency of oscillations, it leads by the same angle over the wall temperature. For large value of ω, we see that the rate of heat transfer from the wall to the fluid leads by an angle 45°.

References

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